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Periodic Solutions for a Class of Epidemic Equations

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Periodic solutions for a class of delay integral equations modeling epidemics are shown to bifurcate from the identically zero solution when a certain parameter exceeds a threshold. The equations are a special case of a general model proposed by Hoppensteadt and Waltman [3]. A global bifurcation theorem of Roger Nussbaum [5] is the main tool.

1. INTRODUCTION

It has been observed that certain infectious diseases have periodic incidence in human populations [2, 4]. Several authors have proposed mathematical models to explain the periodic outbreaks; we mention those of Hethcote [2] and Cooke and Kaplan [1]. Both models assume only two classes of individuals, susceptibles and infectives. These models ignore the possible effects of an incubation period prior to becoming infectious and effects due to a period of temporary immunity following recovery. In this paper we consider a special case of a model proposed by Hoppensteadt and Waltman [3] which includes an incubation period and a period of temporary immunity. It will be shown that nontrivial periodic solutions of the delay-integral equations arising from this model occur when a certain threshold has been exceeded. This threshold involves the contact rate of susceptibles with infectives, assumed to be a periodic function of time, the duration of the period of incubation, and the duration of infectiousness but not the duration of temporary immunity. For the special case considered, this answers a question suggested in Waltman [8, chapt. 5]. If the threshold is not exceeded no periodic incidence is possible, in fact, the disease dies out exponentially fast. The periodic solutions are shown to bifurcate from the zero solution at the critical value of the threshold parameter. This analysis is then extended to include possible further subdivision of the population into disjoint classes. The main tool used in this analysis is a global bifurcation theorem of Roger Nussbaum [5]. This theorem was used by Nussbaum in a similar way for a two class model in [5]. An alternate approach for the two class model appears in the author's Ph. D. thesis [7]. The work in this paper represents an

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extension of results obtained in the author's Ph. D. thesis and the author wishes to express his appreciation to Professor Paul Waltman for pointing out these interesting problems.

2. The Model

We consider a population, whose size is assumed constant in time, partitioned into four disjoint classes: Those susceptible to the disease but not yet exposed, those exposed to the disease who are assumed to be incubating the disease and not yet capable of transmitting it, those infected with the disease who are capable of transmitting it, and those individuals who have recovered, cannot transmit the disease, and who are temporarily immune to the disease. We denote these classes by the letters S, E, I, R respectively; it will be convenient also to use these letters to denote the proportion of the population in each of these classes. Thus, $S + E + I + R = 1$. The course that an individual takes is represented symbolically by $S \to E \to I \to R \to S$. We assume that the infection spreads according to the following rules:

(i) The rate of exposure of susceptibles to infectives at time t is given by $a(t)\, S(t)\, I(t)$, where $a(t)$, the contact rate, is assumed to be ω-periodic in t and positive.

(ii) An individual exposed at time $t - \tau_1$ becomes infective at time t. Thus the incubation period is the time spent in the class E and is precisely τ_1 units of time.

(iii) An infectious individual remains so for precisely τ_2 units of time after which he enters the class R.

(iv) A recovered individual has immunity for τ_3 units of time after which he again becomes susceptible.

We point out that the contact rate $a(t)$ is often assumed to have period one year to account for seasonal effects in weather and times when children are in school.

The equations arising from the model are easily seen to be the following:

$$\begin{aligned} S'(t) &= -a(t)\, S(t)\, I(t) \\ &\quad + a(t - \tau_1 - \tau_2 - \tau_3)\, S(t - \tau_1 - \tau_2 - \tau_3)\, I(t - \tau_1 - \tau_2 - \tau_3) \end{aligned} \tag{1.1}$$

$$E'(t) = a(t)\, S(t)\, I(t) - a(t - \tau_1)\, I(t - \tau_1)\, S(t - \tau_1) \tag{1.2}$$

$$\begin{aligned} I'(t) &= a(t - \tau_1)\, I(t - \tau_1)\, S(t - \tau_1) \\ &\quad - a(t - \tau_1 - \tau_2)\, I(t - \tau_1 - \tau_2)\, S(t - \tau_1 - \tau_2) \end{aligned} \tag{1.3}$$

$$R'(t) = a(t-\tau_1-\tau_2)\, I(t-\tau_1-\tau_2)\, S(t-\tau_1-\tau_2)$$
$$- a(t-\tau_1-\tau_2-\tau_3)\, I(t-\tau_1-\tau_2-\tau_3)\, S(t-\tau_1-\tau_2-\tau_3). \tag{1.4}$$

These equations can be written in integral form as

$$S(t) = 1 - \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(\eta)\, S(\eta)\, I(\eta)\, d\eta \tag{1.5}$$

$$E(t) = \int_{t-\tau_1}^{t} a(\eta)\, S(\eta)\, I(\eta)\, d\eta \tag{1.6}$$

$$I(t) = \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(\eta)\, S(\eta)\, I(\eta)\, d\eta \tag{1.7}$$

$$R(t) = \int_{t-\tau_1-\tau_2-\tau_3}^{t-\tau_1-\tau_2} a(\eta)\, S(\eta)\, I(\eta)\, d\eta. \tag{1.8}$$

To see that the constants of integration have been correctly chosen it suffices to notice that the integral in (1.5), for instance, represents the sum of those individuals who have left the susceptible class in the time period $[t-\tau_1-\tau_2-\tau_3, t]$ and thus cannot return to the class S until after time t. Hence this integral represents all those in the classes E, I, and R at time t.

The equations (1.5)–(1.8) can also be obtained from the more general model of Hoppensteadt and Waltman [3] by choosing $\rho_1 =$ constant, $\rho_2 = 0$, $r(t)$ ω-periodic, and suppressing the initial conditions. This amounts to looking at the asymptotic form of their equations when $\rho_1 =$ constant and $\rho_2 = 0$.

Clearly, it is sufficient to consider only (1.5) and (1.7). With the change of variables $x_1 = 1 - S$, $x_2 = I$, these equations become:

$$x_1(t) = \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s) f(x_1(s), x_2(s))\, ds \tag{1.9}$$

$$x_2(t) = \int_{t\ \ \tau_1-\tau_2}^{t-\tau_1} a(s) f(x_1(s), x_2(s))\, ds \tag{1.10}$$

where

$$\begin{aligned} f(x_1, x_2) &= x_2(1-x_1), \qquad & 0 \leqslant x_1 \leqslant 1 \\ &= 0, & x_1 \geqslant 1. \end{aligned}$$

Equations (1.9) and (1.10) describe the model to be considered in this paper.

Before proceeding to the analysis of equations (1.9) and (1.10) we point out how these equations can serve as a model when more than four classes are present. Assume a constant population is partitioned into n disjoint classes

C_1, C_2,..., C_n where the progress of an individual is described symbolically by $C_1 \to C_2 \to C_3 \to \cdots \to C_n \to C_1$. The following rules are assumed to describe the dynamics:

(i) An idividual in the class C_i remains there precisely τ_i units of time, $2 \leqslant i \leqslant n$.

(ii) Individuals move from class C_1 to C_2 at the rate $a(t)\, C_1(t)\, C_j(t)$ where $a(t)$ is ω-periodic and j is a fixed integer $2 \leqslant j \leqslant n$.

In the previous epidemic model, $n = 4$ and $j = 3$. In the model of Cooke and Kaplan [1] $n = 2$, $j = 1$. As before we have $C_1 + C_2 + \cdots + C_n = 1$ and the equations are easily seen to be

$$\begin{aligned}
C_1(t) &= 1 - \int_{t-\tau_2-\tau_3-\cdots-\tau_n}^{t} a(s)\, C_1(s)\, C_j(s)\, ds \\
C_2(t) &= \int_{t-\tau_2}^{t} a(s)\, C_1(s)\, C_j(s)\, ds \\
{}^{n,j}&\vdots \\
C_j(t) &= \int_{t-\tau_2-\cdots-\tau_{j-1}-\tau_j}^{t-\tau_2-\cdots-\tau_{j-1}} a(s)\, C_1(s)\, C_j(s)\, ds \\
&\vdots \\
C_n(t) &= \int_{t-\tau_2-\cdots-\tau_n}^{t-\tau_2-\cdots-\tau_{n-1}} a(s)\, C_1(s)\, C_j(s)\, ds.
\end{aligned} \tag{1.11}$$

If, as before, we let $x_1 = 1 - C_1$ and $x_2 = C_j$ we obtain the two equations

$$x_1(t) = \int_{t-\tau_2-\cdots-\tau_n}^{t} a(s)\, x_2(s)\, (1 - x_1(s))\, ds \tag{1.12}$$

$$x_2(t) = \int_{t-\tau_2-\cdots-\tau_j}^{t-\tau_2-\cdots-\tau_{j-1}} a(s)\, x_2(s)\, (1 - x_1(s))\, ds. \tag{1.13}$$

These equations differ from (1.9) and (1.10) only in the appearance of more delays which do not affect the mathematical analysis. Our theorems will be stated in terms of the equations (1.9) and (1.10) though we will point out any significant differences which arise from the appearance of more delays in (1.12) and (1.13). Finally, the law of mass action has been used to obtain the particular f in (1.9) and (1.10). If $a(t)\, I(t)\, S(t)$ is replaced by $a(t)\, h(S(t), I(t))$, the transformation $x_1 = 1 - S$, $x_2 = I$ gives

$$\begin{aligned}
f(x_1, x_2) &= h(1 - x_1, x_2), & 0 \leqslant x_1 \leqslant 1 \\
&= 0, & x_1 > 1.
\end{aligned}$$

From a biological standpoint it is reasonable to assume that $h(S, 0) = h(0, I) = 0$ for $I \geqslant 0$, $S \geqslant 0$, $h(S, I) > 0$ for $S > 0$ and $I > 0$, and h increases as $S(I)$ increases. From a mathematical standpoint, the linearization of f about $(x_1, x_2) = (0, 0)$ will be important. An easy calculation, given the above assumptions on h, gives $(\partial f/\partial x_1)(0, 0)\, x_1 + (\partial f/\partial x_2)(0, 0)\, x_2 = (\partial h/\partial I)(1, 0)\, x_2$. The fact that the linearization of f has no x_1-term greatly simplifies the analysis. In addition we assume $(\partial h/\partial I)(1, 0) = 1$. This is no loss in mathematical generality since the constant $(\partial h/\partial I)(1, 0)$ can be absorbed in the contact rate $a(t)$. Equations (1.9) and (1.10) will be treated where $f(x_1, x_2)$ need note be that dictated by mass action. The above remarks motivate the assumptions to be placed on f is the next section.

3. Main Results

Before stating our main results we list the assumptions to be used in the theorems. The proofs appear in the following section.

(H1) $a(t)$ is a continuous, ω-periodic, positive function defined on $\mathbb{R}$.

(H2) $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $f(x_1, 0) \equiv 0$ for $x_1 \geqslant 0$, $f(x_1, x_2) = 0$ if $x_1 \geqslant 1$, and $f(x_1, x_2)$ is nondecreasing in x_2 for fixed x_1. In addition assume f is continuously differentiable on $[0, 1] \times [0, 1]$ and $(\partial f/\partial x_1)(0, 0) = 0$, $(\partial f/\partial x_2)(0, 0) = 1$. Also $f(x_1, x_2) > 0$ for $0 < x_2$, $x_1 < 1$.

(H3) The constants τ_1, τ_2, τ_3 are nonnegative real numbers.

Theorem 3.1. *Assume* (H1), (H2), *and* (H3) *hold. Then there exists a critical value* $\tau_2^* > 0$ *of the parameter* τ_2 *depending only on the function* $a(t)$ *and the parameter* τ_1 *such that for* $\tau_2 > \tau_2^*$, (1.9) *and* (1.10) *have at least one nonzero, ω-periodic solution* $(x_1(t), x_2(t))$ *satisfying* $0 < x_2(t) \leqslant x_1(t) < 1$ *for* $t \in \mathbb{R}$. *Moreover, the nonzero ω-periodic solutions bifurcate from the zero solution* $(x_1, x_2) = (0, 0)$ *at* $\tau_2 = \tau_2^*$.

If in addition f satisfies:

(H4) $f(x_1, x_2) \leqslant x_2$ for $0 \leqslant x_2 \leqslant x_1 \leqslant 1$,

then the behavior of solutions for $\tau_2 < \tau_2^*$ is described in the next result.

Theorem 3.2. *Assume that* (H1)–H(4) *hold. Then there are no nonzero ω-periodic solutions of* (1.9), (1.10) *if* $\tau_2 < \tau_2^*$. *In fact, if* $\tau_2 < \tau_2^*$ *and* $(x_1(t), x_2(t)) \geqslant (0, 0)$ *satisfies* (1.9) *and* (1.10) *for* $t \geqslant 0$ *with* $(x_1(t), x_2(t)) = (\phi_1(t), \phi_2(t)) \geqslant (0, 0)$ *for* $t \in [-\tau_1 - \tau_2 - \tau_3, 0]$, *then there exists* M_1, M_2, $\eta > 0$ *such that* $0 \leqslant x_i(t) \leqslant M_i e^{-\eta t}$ $(i = 1, 2)$.

Recalling that $S = 1 - x_1$ and $I = x_2$, Theorems 3.1 and 3.2 imply that periodic incidence occurs if the duration of infectiousness exceeds τ_2^*. This critical value depends on the contact rate, $a(t)$, and the length of the period of incubation, τ_1, but not on the length of the period of immunity, τ_3.

We remark that the assumption that $a(t)$ be positive for all t can be weakened to allow $a(t)$ to vanish, though not on a set of positive measure. Also it should be pointed out that the differentiability assumptions on f are stronger than necessary. We use differential inequality arguments to obtain the estimates $0 < x_2(t) \leq x_1(t) < 1$. These argulents require the uniqueness of solutions of certain ordinary differential equations and it is convenient for this to assume $f \in C^1([0, 1] \times [0, 1])$. If the strict inequalities of the estimates are not required then it suffices to assume $f \in C^1([0, \delta] \times [0, \delta])$ for some $\delta > 0$.

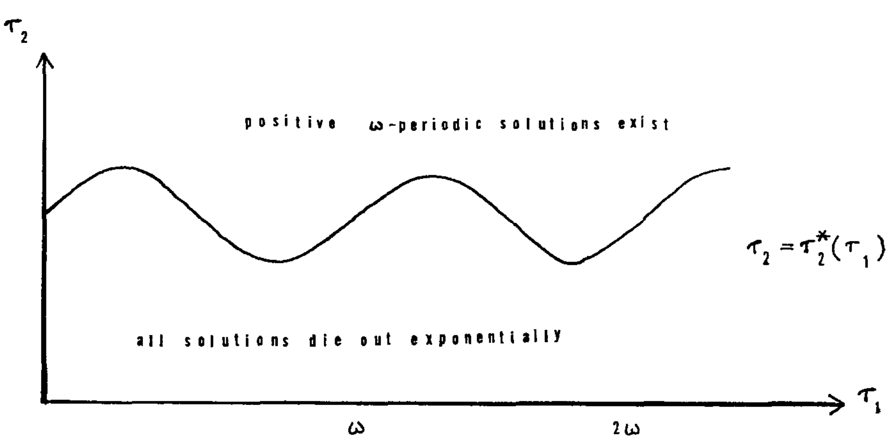
When $\tau_2 > \tau_2^*$, there are solutions $S(t)$, $I(t)$, $E(t)$, and $R(t)$ which are ω-periodic and the estimates of Theorem 3.1 imply that $0 < S(t) < 1$, $0 < I(t) < 1$, $0 < E(t) < 1$, and $0 < R(t) < 1$ as expected. If $\tau_2 < \tau_2^*$, then $S(t) \to 1$, $I(t) \to 0$, $E(t) \to 0$, and $R(t) \to 0$ exponentially fast. Thus the infection dies out.

The results of Theorems 3.1 and 3.2 apply with only one change to the equations (1.12) and (1.13). In this case the critical parameter is τ_j^* and τ_j^* depends on the contact rate $a(t)$ and the sum $\tau_2 + \cdots + \tau_{j-1}$, but is independent of $\tau_{j+1}, \ldots, \tau_n$.

In Theorem 3.1, τ_1 is fixed and it is shown there is a $\tau_2^* = \tau_2^*(\tau_1)$ such that periodic solutions of (1.9), (1.10) bifurcate from the zero solution at τ_2^*. The next result shows how τ_2^* depends on τ_1.

THEOREM 3.3. *Assuming* (H1)–(H3) *hold*, $\tau_2^*: [0, \infty) \to (0, \infty)$ *is a continuous ω-periodic function of* τ_1 *satisfying the estimate* $[\sup a(t)]^{-1} \leq \tau_2^*(\tau_1) \leq [\inf a(t)]^{-1}$.

The results of Theorems 3.1, 3.2, and 3.3 are summarized in the figure below.



Theorem 3.3 can be viewed as a stability result: if $\tau_2 > \tau_2{}^*(\tau_1)$ ($\tau_2 < \tau_2{}^*(\tau_1)$) and if τ_1 is perturbed slightly to $\tau_1{}'$ and τ_2 is perturbed to $\tau_2{}'$ then $\tau_2{}' > \tau_2{}^*(\tau_1{}')$ ($\tau_2{}' < \tau_2{}^*(\tau_1{}')$) provided $|\tau_1 - \tau_1{}'| + |\tau_2 - \tau_2{}'|$ is sufficiently small. Thus the qualitative behavior of solutions is stable to small perturbations in τ_1 and τ_2. We also point out that if $a(t) \equiv a$ is constant then $\tau_2{}^*(\tau_1) \equiv a^{-1}$. In this case the periodic solutions are constant as well. In certain other special situations $\tau_2{}^*$ can be calculated exactly. It will follow from the proof of Theorem 3.3, as in [7], that if for some positive integer n, $n \int_0^\omega a(t)\, dt = 1$, then $\tau_2{}^*(\tau_1) \equiv (1/\omega \int_0^\omega a(t)\, dt)^{-1} = n\omega$. Notice, in this case, that $\tau_2{}^*$ is independent of τ_1. The author conjectures that, in general, $[1/\omega \int_0^\omega a(s)\, ds]^{-1}$ is the mean value of $\tau_2{}^*$ but has been unable to establish this.

In the case that equations (1.12) and (1.13) are being considered, $\tau_j{}^*$ depends on $l = \tau_1 + \tau_2 + \cdots + \tau_{j-1}$ and Theorem 3.3 is changed to read that $\tau_j{}^*$ is a continuous, positive, ω-periodic function of $l \in [0, \infty)$. The above remarks also apply in this case.

Several interesting problems concerning (1.9) and (1.10) remain unsolved. The question of uniqueness of the nonzero ω-periodic solutions for fixed $\tau_1 \geqslant 0$, $\tau_3 \geqslant 0$ and $\tau_2 > \tau_2{}^*(\tau_1)$ is one. Another very important problem is whether the ω-periodic solutions are stable.

4. Proofs of the Theorems

To establish the existence and bifurcation of ω-periodic solutions of (1.9) and (1.10) we employ a global bifurcation theorem of Roger Nussbaum [5, Theorem 1]. Before stating this theorem we require some notation. Let $J = (0, \infty)$ and K be a cone in the real Banach space X. Consider a map $F: K \times J \to K$ satisfying $F(0, \tau) = 0$ and the following:

A. $F: K \times J \to K$ is a continuous map taking bounded sets into precompact sets. If $F(x_k, \tau_k) = x_k$ for a sequence such that $x_k \in K - \{0\}$ and $\tau_k \to 0$, then it follows that $\lim_{k\to\infty} \| x_k \| = +\infty$.

B. For each $\tau \in J$ there exists a compact linear operator L_τ leaving K invariant with the property that for any compact interval $[c, d] \subset J$,

$$\lim_{\|x\|\to 0} (\| x \|^{-1} (\| F(x, \tau) - L_\tau(x) \|) = 0$$

uniformly in $\tau \in [c, d]$. The map $\tau \mapsto L_\tau$ is continuous with respect to the operator norm.

C. There exists a countable family $\Lambda \subset J$ with no finite accumulation point such that $x \neq L_\tau x$ for $x \in K - \{0\}$ and $\tau \notin \Lambda$.

In the statement of the theorem to follow, $r(A)$ denotes the spectral radius of the linear operator A.

THEOREM (Nussbaum). *Assume that* A, B, *and* C *hold. Suppose that for some* $\tau_0 \in \Lambda$, $r(L_\tau) > 1$ *for* $\tau > \tau_0$ *and* τ *near* τ_0 *and* $r(L_\tau) < 1$ *for* $\tau < \tau_0$ *and* τ *near* τ_0 *(or vice versa). Let* S *denote the closure in* $K \times J$ *of* $\{(x, \tau) \in K \times J : x \neq 0$ *and* $F(x, \tau) = x\}$ *and* S_0 *denote the maximal connected component of* S *which contains* $(0, \tau_0)$. *Then it follows that* S_0 *is nonempty and either* S_0 *is unbounded or* S_0 *contains a point* $(0, \tau_1)$ *with* $\tau_1 \in \Lambda$, $\tau_1 \neq \tau_0$. *If* S_0 *is bounded and* Λ_0 *denotes the finite set of* $\tau \in \Lambda$ *such that* $(0, \tau) \in S_0$, *it follows that* $\sum_{\tau \in \Lambda_0} \Delta(\tau) = 0$.

In the above, $\Delta(\tau) = \lim_{\tau \to \tau_0^+} i_k(F_\tau, B_{\rho(\tau)}) - \lim_{\tau \to \tau_0^-} i_k(F_\tau, B_{\rho(\tau)})$, where $i_k(F_\tau, B_{\rho(\tau)})$ denotes the fixed point index of F_τ on $B_{\rho(\tau)} = \{x \in K : \|x\| < \rho(\tau)\}$ with $\rho(\tau)$ chosen sufficiently small $(\tau \notin \Lambda)$ so there are no fixed points of $F_\tau = F(\cdot, \tau)$ on $\bar{B}_{\rho(\tau)} - \{0\}$. In our problem it will turn out that S_0 is unbounded so the last sentence of the theorem does not apply.

Proof of Theorem 3.1. Let E_1 be the Banach space of ω-periodic continuous real-valued functions with supremum norm $(\|\cdot\|_\infty)$. Let K_1 be the cone of nonnegative functions in E_1. Define $E_2 = E_1 \times E_1$ with norm $\|(x_1, x_2)\| = \|x_1\|_\infty + \|x_2\|_\infty$ and let $K_2 = K_1 \times K_1$ be the cone in E_2 of functions with nonnegative components. Fix τ_1 and τ_3 nonnegative and define the operator F on $K_2 \times J$ by $F((x_1, x_2), \tau_2) = (\tilde{x}_1, \tilde{x}_2)$, where

$$\tilde{x}_1(t) = \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s) f(x_1(s), x_2(s))\, ds$$

$$\tilde{x}_2(t) = \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s) f(x_1(s), x_2(s))\, ds.$$

It is easily seen that $(\tilde{x}_1, \tilde{x}_2) \in K_2$. It is also routine (see [5] or [6] for a similar calculation) to check that $F: K_2 \times J \to K_2$ is continuous and maps bounded sets into precompact sets. Notice that $F((0, 0), \tau_2) = (0, 0)$. To check that A of the theorem holds we show that there does not exist a sequence $\{((x_1^k, x_2^k), \tau_2^k)\} \subseteq K_2 \times J$ such that $(x_1^k, x_2^k) \neq (0, 0)$, $F((x_1^k, x_2^k), \tau_2^k) = (x_1^k, x_2^k)$, and $\tau_k \to 0$. If such a sequence existed then clearly $\{(x_1^k, x_2^k)\}$ would be a bounded sequence since

$$0 \leqslant x_2^k(t) \leqslant x_1^k(t) \leqslant \int_{t-\tau_1-\tau_2^k-\tau_3}^{t} a(s) f(x_1^k(s), x_1^k(s))\, ds$$
$$\leqslant K \int_{t-\tau_1-\tau_2^k-\tau_3}^{t} a(s)\, ds$$

where $K = \sup_{0 \leqslant x_1 \leqslant 1} f(x_1, x_1)$. In the above estimate we used the fact that f

is nondecreasing in x_2 for fixed x_1, and the fact that $K < \infty$. But then $(x_1{}^k, x_2{}^k) \to (0, 0)$ since

$$x_2{}^k(t) \leqslant K \int_{t-\tau_1-\tau_2{}^k}^{t-\tau_1} a(s)\, ds \to 0 \qquad \text{as} \qquad k \to \infty$$

and so

$$x_1{}^k(t) = \int_{t-\tau_1-\tau_2^k-\tau_3}^{t} a(s) f(x_1{}^k(s), x_2{}^k(s))\, ds \to 0 \qquad \text{as} \qquad k \to \infty$$

since $f(x_1, 0) \equiv 0$ and f is continuous. Now applying (H2) we may write

$$x_2{}^k(t) = \int_{t-\tau_1-\tau_2{}^k}^{t-\tau_1} a(s)\, x_2{}^k(s)\, ds + o(\|(x_1{}^k, x_2{}^k)\|) \qquad \text{as} \qquad k \to \infty.$$

Hence $\| x_2{}^k \|_\infty \leqslant A\tau_2{}^k \| x^k \|_\infty + o(\| x_1{}^k \|_\infty + \| x_2{}^k \|_\infty)$, where $A = \sup a(t)$. Similarly

$$\begin{aligned} \| x_1{}^k \| &\leqslant \sup_t \int_{t-\tau_1-\tau_2{}^k-\tau_3}^{t} a(s)\, ds\, \| x_2{}^k \|_\infty + o(\| x_1{}^k \|_\infty + \| x_2{}^k \|_\infty) \\ &\leqslant P \| x_2{}^k \|_\infty + o(\| x_1{}^k \|_\infty + \| x_2{}^k \|_\infty) \end{aligned}$$

where

$$P = \sup_{t,k} \int_{t-\tau_1-\tau_2{}^k-\tau_3}^{t} a(s)\, ds.$$

Replacing $\| x_2{}^k \|_\infty$ in this last estimate by the right hand side of the previous estimate and observing that $\| x_2{}^k \|_\infty \leqslant \| x_1{}^k \|_\infty$ we have

$$\| x_1{}^k \| \leqslant AP\tau_2{}^k \| x_2{}^k \|_\infty + o(\| x_1{}^k \|_\infty)$$

or $1 \leqslant AP\tau_2{}^k + o(1) \to 0$ as $k \to \infty$. This contradiction shows no such sequence exists and thus A holds. B is easily seen to hold with L_{τ_2} given by

$$[L_{\tau_2}(h_1, h_2)]\,(t) = \left(\int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s)\, h_2(s)\, ds, \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s)\, h_2(s)\, ds \right).$$

From the expression for L_{τ_2} it is evident that the map $\tau_2 \mapsto L_{\tau_2}$ is continuous with respect to the operator norm. That L_{τ_2} is a compact, positive linear operator is readily seen. We now require certain spectral properties of L_{τ_2}. Define the linear operator T_{τ_2} on E_1 by

$$(T_{\tau_2} h)\,(t) = \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s)\, h(s)\, ds.$$

We claim that the spectral radius of L_{τ_2} is identical to that of T_{τ_2}. To see this, notice that both operators are compact (though on different spaces) so if $\lambda \in \sigma(L_{\tau_2})$ (the spectrum of L_{τ_2}) then there is a corresponding eigenvector $(h_1, h_2) \neq (0, 0)$. It can readily be seen that $h_2 \neq 0$ and $T_{\tau_2} h_2 = \lambda h_2$. Thus $\lambda \in \sigma(T_{\tau_2})$. Conversely, if $\lambda \in \sigma(T_{\tau_2})$ with corresponding eigenvector h_2, choose $h_1(t)\ \lambda^{-1} \int_{t-\tau_1-\tau_2-\tau_3}^{t-\tau_2} a(s) \times h_2(s)\, ds$. Then (h_1, h_2) is an eigenvector for L_{τ_2} corresponding to λ. Hence the claim has been established. In [5 or 6] it is shown that $r(T_{\tau_2})$ ($r(A)$ is the spectral radius of A) is a strictly increasing continuous function of τ_2 satisfying the estimate:

$$\min_t \int_{t-\tau_2}^{t} a(s)\, ds \leqslant r(T_{\tau_2}) \leqslant \max_t \int_{t-\tau_2}^{t} a(s)\, ds.$$

In particular $(rT_{\tau_2}) \to 0$ as $\tau_2 \to 0$ and $r(T_{\tau_2}) \to \infty$ as $\tau_2 \to \infty$. Thus there is a unique $\tau_2{}^*$ for which $r(T_{\tau_2{}^*}) = 1$ and $r(T_{\tau_2}) < 1$ ($r(T_{\tau_2}) > 1$) if $\tau_2 < \tau_2{}^*$ ($\tau_2 > \tau_2{}^*$). Since $r(T_{\tau_2}) = r(L_{\tau_2})$ we see that $r(L_{\tau_2})$ depends only on τ_1, $a(t)$, and τ_2 but not on τ_3.

In [5, 6] it was shown that T_{τ_2} has precisely one unit norm eigenvector in K_1, the corresponding eigenvalue being $r(T_{\tau_2})$. The argument showing $\sigma(T_{\tau_2}) = \sigma(L_{\tau_2})$ shows this must be the case for L_{τ_2} as well. Hence $\Lambda = \{\tau_2{}^*\}$ and C is satisfied. We can now apply Nussbaum's theorem. Let S be the closure in $K_2 \times J$ of $\{((x_1, x_2), \tau_2) \in K_2 \times J: (x_1, x_2) \neq (0, 0)$ and $F((x_1, x_2), \tau_2) = (x_1, x_2)\}$ and S_0 the maximal connected component of S which contains $((0, 0), \tau_2{}^*)$. Then $S_0 \neq \phi$ and S_0 is unbounded ($\Lambda = \{\tau_2{}^*\}$).

Recall that if $F((x_1, x_2), \tau_2) = (x_1, x_2)$ then $0 \leqslant x_2(t)\ x_1(t) \leqslant K \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s)\, ds$. Hence if $I \subseteq J$ is bounded and $((x_1, x_2), \tau_2) \in S_0$ with $\tau_2 \in I$ then $\|(x_1, x_2)\| \leqslant M$ for $M = 2K \sup_{t\in[0,\omega]/\tau_2\in I} \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s)\, ds$. Thus if S_0 is unbounded and connected it follows that for each $\tau_2 > \tau_2{}^*$ there exists $(x_1, x_2) \in K_2$ such that $((x_1, x_2), \tau_2) \in S_0$, i.e., $F((x_1, x_2), \tau_2) = (x_1, x_2)$. Also $(x_1, x_2) \neq (0, 0)$ since $((0, 0), \tau_2) \notin S$ if $\tau_2 \neq \tau_2{}^*$. Since fixed points of $F_{\tau'} = F_{\tau_2} = F(\cdot, \tau_2)$ correspond to ω-periodic solutions of (1.9) and (1.10) we have established Theorem 3.1 except for the estimates $0 < x_2(t) \leqslant x_1(t) < 1$.

Suppose that $(x_1, x_2) \in K_2 - \{(0, 0)\}$ is an ω-periodic solution of (1.9), (1.10). We want to show $0 < x_2(t) \leqslant x_1(t) < 1$. There are two cases: $\tau_1 = 0$ and $\tau_1 > 0$. Suppose $\tau_1 = 0$. If $x_2(t_0) = 0$ for some t_0 then since $x_2'(t) \leqslant a(t) f(x_1(t), x_2(t))$ ($f(x_1, 0) \equiv 0$) standard differential inequality arguments give $x_2(t) = 0$ for $t \geqslant t_0$ so $x_2 = 0$. Clearly this implies $x_1 = 0$ and a contradiction. Hence $x_2(t) > 0$. Clearly $x_2(t) \leqslant x_1(t)$. Notice that certainly $x_1(t) < 1$ for some values of t, say t_1. Now $x_1(t)$ satisfies the differential inequality $x''(t) \leqslant a(t) f(x_1(t), x_2(t))$ with $x_1(t_1) < 1$ and the solution $u(t) \equiv 1$ is unique for the problem $u'(t) = a(t) f(u(t), x_2(t))$, $u(t_2) = 1$, $t_2 \in \mathbb{R}$. Thus if $v(t)$ satisfies $v'(t) = a(t) f(v(t), x_2(t))$, $v(t_1) = x_1(t_1)$, then $x_1(t) \leqslant v(t) < u(t) = 1$, $t \geqslant t_1$. Thus $x_1(t) < 1$.

Now if $\tau_1 > 0$, suppose $x_1(t_0) = 0$ for some t_0. Then (H1) and (H2) imply that $f(x_1(s), x_2(s)) \equiv 0$ for $s \in [t_0 - \tau_1 - \tau_2 - \tau_3, t_0]$. But then from the equation for x_2 it follows that $x_2(t) \equiv 0$ for $t \in [t_0 - \tau_3, t_0 + \tau_1]$. This in turn implies $f(x_1(s), x_2(s)) \equiv 0$ for $s \in [t_0 - \tau_1 - \tau_2 - \tau_3, t_0 + \tau_1]$. But then $x_2(t) \equiv 0$ for $t \in [t_0 - \tau_3, t_0 + 2\tau_1]$. Continuing in this manner it is apparent that $x_2(t) = 0$ for $t \geqslant t_0 - \tau_3$. Hence $x_2 = 0$ and so $x_1 = 0$, a contradiction. Thus $x_1(t) > 0$. Exactly as before, $x_1(t) < 1$. The estimate $x_2(t) \leqslant x_1(t)$ is trivial so all that remains is to show $x_2(t) > 0$. Since $f(x_1, x_2) > 0$ for $0 < x_2$, $x_1 < 1$, the only way f can vanish is for x_2 to vanish. Making use of this fact and the equation for x_2 it is easily shown that $x_2(t) > 0$.

Proof of Theorem 3.2. Let $\tau_2 < \tau_2^*$ and let $T_{\tau_2}: E_1 \to E_1$ be the linear operator defined in the proof of Theorem 3.1. Since $r(T_{\tau_2}) = \lim_{n\to\infty} \| T_{\tau_2}^n \|^{1/n} < 1$, there exists N, a positive integer such that $\| T_{\tau_2}^N \| = e^{-\gamma}$ for some $\gamma > 0$. It is easily seen from the definition of T_{τ_2} that

$$\| T_{\tau_2}^N \| = \sup_{0 \leqslant t \leqslant \omega} \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s_1) \int_{s_1-\tau_1-\tau_2}^{s_1-\tau_1} a(s_2) \cdots \int_{s_{N-1}-\tau_1-\tau_2}^{s_{N-1}-\tau_1} a(s_N)\, ds_N \cdots ds_1 .$$

Now suppose $(x_1(t), x_2(t)) \geqslant (0, 0)$ satisfies (1.9) and (1.10) for $t \geqslant 0$ with $(x_1(t), x_2(t)) = (\phi_1(t), \phi_2(t))$ for $t \in [-\tau_1 - \tau_2 - \tau_3, 0)$ where $\phi_i(t) \geqslant 0$, $i = 1, 2$. We have

$$\left.\begin{aligned} x_1(t) &= \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s) f(x_1(s), x_2(s))\, ds \\ x_2(t) &= \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s) f(x_1(s), x_2(s))\, ds \end{aligned}\right\} \quad \text{for } t \geqslant 0.$$

Clearly $x_2(t) \leqslant x_1(t)$ for $t \geqslant 0$.

It follows that for $t \geqslant \tau_1 + \tau_2 + \tau_3$

$$x_1(t) \leqslant \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s) f(x_1(s), x_1(s))\, ds \leqslant K \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s)\, ds$$

where $K = \sup_{x_1 \geqslant 0} f(x_1, x_1)$ and we have used the fact that f is nondecreasing in its second variable. In particular we conclude that both x_1 and x_2 are bounded on $t \geqslant -\tau_1 - \tau_2 - \tau_3$. Let $M = \sup x_2(t)$. Applying (H4) we obtain

$$x_2(t) = \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s) f(x_1(s), s_2(s)\, ds \leqslant \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s)\, x_2(s)\, ds, \tag{1.14}$$

so $x_2(t) \leqslant M \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s)\, ds$ for $t \geqslant 0$. Combining this last inequality with (1.14)

$$x_2(t) \leqslant M \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s_1) \int_{s-\tau_1-\tau_2}^{s_1-\tau_1} a(s_2)\, ds_2\, ds_1 , \qquad \text{for } t \geqslant \tau_1 + \tau_2 .$$

Continuing in this fashion we arrive at

$$x_2(t) \leqslant M \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s_1) \int_{s_1-\tau_1-\tau_2}^{s-\tau_1} a(s_2) \cdots \int_{s_{N-1}-\tau_1-\tau_2}^{s_{N-1}-\tau} a(s_N)\, ds_N \cdots ds_1 \leqslant Me^{-\gamma}$$

$$\text{if} \qquad t \geqslant (N-1)(\tau_1 + \tau_2), \qquad \text{for some } \gamma > 0.$$

We may now repeat the above argument k times to obtain $x_2(t) \leqslant Me^{-k\gamma}$ if $t \geqslant (kN - 1)(\tau_1 + \tau_2)$. An induction argument shows this last estimate holds for all positive integers k. It is now easy to see that for some $\eta > 0$, $M_2 > 0$, $x_2(t) \leqslant M_2 e^{-\eta t}$. Since

$$x_1(t) \leqslant \int_{t-\tau_1-\tau_2-\tau_3}^{t} a(s)\, x_2(s)\, ds$$

the corresponding estimate for x_1 also holds.

Proof of Theorem 3.3. Recall from the proof of Theorem 3.1 that $\tau_2{}^*$ is the ***unique*** value of τ_2 for which the spectral radius of $T_{\tau_1,\tau_2}: E_1 \to E_1$ is 1, where $(T_{\tau_1,\tau_2} h)(t) = \int_{t-\tau_1-\tau_2}^{t-\tau_1} a(s)\, h(s)\, ds$. Denote by $r(\tau_1, \tau_2)$ the spectral radius of T_{τ_1,τ_2}. Notice that $T_{\tau_1+\omega,\tau_2} = T_{\tau_1,\tau_2}$ so $r(\tau_1 + \omega, \tau_2) = r(\tau_1, \tau_2)$. This implies that $\tau_2{}^*(\tau_1 + \omega) = \tau_2{}^*(\tau_1)$. The fact that $\tau_2{}^*$ is continuous in τ_1 will follow from the fact that $r(\tau_1, \tau_2)$ is continuous on $[0, \infty) \times (0, \infty)$ and satisfies the estimate: $\min_t \int_{t-\tau_2}^{t} a(s)\, ds \leqslant r(\tau_1, \tau_2) \leqslant \max_t \int_{t-\tau_2}^{t} a(s)\, ds$. The estimate follows as in [5, 6]. To see that r is continuous let $(\tau_1{}^0, \tau_2{}^0) \in [0, \infty) \times (0, \infty)$ and $\{(\tau_1{}^k, \tau_2{}^k)\} \subseteq [0, \infty) \times (0, \infty)$ with $(\tau_1{}^k, \tau_2{}^k) \to (\tau_1{}^0, \tau_2{}^0)$. Let $T_k = T_{\tau_1{}^k,\tau_2{}^k}$, $T_0 = T_{\tau_1{}^0,\tau_2{}^0}$, $r_k = r(\tau_1{}^k, \tau_2{}^k)$, $r_0 = r(\tau_1{}^0, \tau_2{}^0)$.

As pointed out earlier, there exists, for each k, a unique unit norm eigenvector $h_k \in K_1$ for T_k with corresponding eigenvalue r_k: $T_k h_k = r_k h_k$. If $r_k \not\to r_0$ then we can choose a subsequence $r_{k_i} \to r' \neq r_0$ ($\{r_k\}$ is bounded by our estimate). It is easy to see that $\{h_{k_i}\}$ is precompact an we will assume $h_{k_i} \to h_0 \in K_1$, $\| h_0 \| = 1$. Now

$$\| T_0 h_0 - r' h_0 \| \leqslant \| T_0 h_0 - T_0 h_{k_i} \| + \| T_0 h_{k_i} - T_{k_i} h_{k_i} \| + \| r_{k_i} h_{k_i} - r' h_0 \| .$$

Clearly $\| T_0 h_0 - T_0 h_{k_i} \| \to 0$ and $\| r_{k_i} h_{k_i} - r' h_0 \| \to 0$. It is easy to see that

$$\| T_0 - T_{k_i} \| \leqslant A[| \tau_1{}^0 - \tau_1{}^k | + | \tau_1{}^0 + \tau_2{}^0 - \tau_1{}^k - \tau_2{}^k |]$$

where $A = \sup_t a(t)$ so $\| T_0 - T_{k_i} \| \to 0$ as $i \to \infty$. Hence $T_0 h_0 = r' h_0$. But as pointed out earlier, T_0 has exactly one unit norm eigenvector in K_1 and the corresponsing eigenvalue is r_0. This contradiction proves the assertion.

It now follows easily that $\tau_2{}^*$ is continuous since $r(\tau_1, \tau_2{}^*(\tau_1)) = 1$. Let $\tau_1{}^k \to \tau_1{}^0$, $\tau_1{}^0, \tau_1{}^k \in [0, \infty)$ for all k. If $\tau_2{}^*(\tau^k) \not\to \tau_2{}^*(\tau_1{}^0)$ we may assume some subsequence converges: $\tau_2{}^*(\tau_1^{k_i}) \to \tau_2' \neq \tau_2{}^*(\tau_1{}^0)$. Since r is continuous

$r(\tau_1{}^0, \tau_2{}') = 1$. But there is only one value of τ_2 for which $r(\tau_1{}^0, \tau_2) = 1$, namely $\tau_2{}^*(\tau_1{}^0)$. This contradiction establishes the continuity of $\tau_2{}^*$.

The estimate of Theorem 3.3 follows from the earlier estimate of $r(\tau_1, \tau_2)$ since

$$\min_t a(t) \cdot \tau_2{}^*(\tau_1) \leqslant \min_t \int_{t-\tau_2{}^*(\tau_1)}^{t} a(s)\, ds \leqslant 1$$

$$\leqslant \max_t \int_{t-\tau_2{}^*(\tau_i)}^{t} a(s)\, ds \leqslant \max_t a(t) \cdot \tau_2{}^*(\tau_1).$$

It has been pointed out by the referree that the assumption that $f(x_1, x_2)$ be nondecreasing in x_2 for fixed x_1 in (H2) can be avoided. This assumption is used to obtain bounds for $x_1(t)$. Using differential inequality arguments as in the proof of Theorem 3.1, one easily shows that $x_1(t) \leqslant 1$.

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